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Productive Ambiguity in Mathematics

Gianluigi Oliveri
Department FIERI-AGLAIA
University of Palermo
gianluigi.oliveri@unipa.it

ABSTRACT. According to E. Grosholz, there is a phenomenon called 'productive ambiguity' which plays a very important rôle in mathematics, and the sciences, because it is instrumental to the resolution of many open questions. The main task of this paper is that of assessing Grosholz's claim with regard to mathematics.

KEYWORDS: Productive ambiguity, Mathematical realism

1. Productive ambiguity

E. Grosholz in her book, *Representation and Productive Ambiguity in Mathematics*, claims that productive ambiguity plays a very important rôle in mathematics and the sciences, because it allows the interaction of different modes of representation, interaction which, in some cases, brings about the resolution of open questions. But, what does Grosholz mean by 'productive ambiguity'?

From the discussion of the case-studies examined in her book, it turns out that productive ambiguity consists in the possibility of interpreting in different ways the well-formed formulae belonging to the language of a mathematical theory T.[1]

[1] Grosholz (2007), ch.3 §. 3.4, p. 76:

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In actual fact, it seems that, for Grosholz, productive ambiguity is more pervasive than that, because it extends also to some of the diagrams used in the proofs of theorems of T.[2]

With regard to the problem of the ambiguity of diagrams used in mathematical proofs, we must observe that, according to Grosholz, this depends on a certain diagram $\mathfrak{D}$ having, in a given context, both an iconic and a symbolic mode of representation.

For Grosholz, the iconic mode of representation of a diagram $\mathfrak{D}$ consists in the fact that $\mathfrak{D}$ pictures what it represents; whereas the symbolic mode of representation of a diagram $\mathfrak{D}$ consists in representing an entity without picturing it.

An example of such an ambiguous rôle of a diagram is provided by the triangle we draw in the well known argument aimed at showing that in Euclidean geometry the sum of the internal angles of a triangle is 180°.

Indeed, in this case:[3]

> The proof requires the triangle [we draw] both to be an icon of a particular triangle, and to represent symbolically . . . all other . . . triangles.

It is important to notice that, according to Grosholz, the coexistence of both an iconic and a symbolic mode of representation is not strict monopoly of diagrams, but it also affects mathematical notation (language) in general.[4]

> In algebraic geometry, a polynomial equation (set = 0) with n variables refers equally to an infinite set of n-tuples of real numbers, to a geometric curve or surface C, and further to another infinite set, all the polynomials whose values are identically zero on C, an ideal in the larger ring of such polynomials. The controlled and precise ambiguity of the equation is the instrument that allows resources of number theory and of geometry to be combined in the service of problem-solving.

[2] Such an extension is evident in Grosholz's case-study of Galileo's demonstration of projectile motion. See Grosholz (2007), §1.1, pp. 5-16.

[3] Grosholz (2007), §6.1, p. 163.

[4] Grosholz (2007), ch.10, §10.2, p. 262:

> A natural number is either the unit or a multiplicity of units in one number. The representation of such a unified multiplicity is more iconic when the representation itself involves multiplicity: ////// is more iconic than '6' or 'six', because ////// exhibits the multiplicity of the number 6—its multiplicity can be 'read

2. Objections

In evaluating Grosholz's ideas about productive ambiguity, it seems to me that, first, there are formidable obstacles in the way of her attempt to generalize to the entire language of mathematical theories the presence of both an iconic and a symbolic mode of representation. One such obstacle appears as soon as we ask what is the iconic mode of representation of the symbol '$\emptyset$'.

To see this consider the following argument. If a set P_A is a picture of a set A, there has to be an isomorphism, Ψ, between A and P_A. For P_A needs to preserve the form, and the cardinality of A.

From this it follows that a set A is a picture of itself—take the trivial isomorphism induced by the identity map on A—and that, if by 'proper picture (image) of A' we mean a picture I of A such that $I \neq A$, we have that A is an improper picture of itself.

Clearly, given a set A, we can have different pictures of A, because what I have called 'form' of a set A is related to the features of A, and different types of isomorphism preserve different features of A.

However, independently of which are the features of A that remain invariant under a given isomorphism Ψ between A and P_A, what must, in any case, obtain is the existence of a bi-univocal correspondence between A and P_A. This implies that if $A = \emptyset$ then also $P_A = \emptyset$.

From these considerations we have that the empty set is the only picture of itself and that, therefore, there cannot be proper pictures (images) of it.

If we, now, consider that the empty set is not given in space-time, it follows that mental images (which are entities given in space-time) can neither be the

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off' the representation.

Grosholz (2007), Ch. 10, §10.2, pp. 265–6:

> Philosophers of mathematics have not clearly recognized this role [of notation in creating models and precipitating nomological machines] because they have been so fixed on symbolic representation and so inattentive to iconic representation, and the iconic (and indexical) aspects of symbolic representations. This has also led them to posit an artificial disjunction between syntax, semantics, and pragmatics. Iconic representations need not be pictures of things with shape, though of course they often are; they may also be representations that exhibit the orderliness that makes something what it is.

empty set nor, for what I have argued above, can be proper pictures of it.

A moment's reflection on what has been argued so far makes us realize that, since we can neither produce mental images of the empty set nor can be acquainted with it—because the empty set is not given in space-time—the only access we have to the empty set is through definitions such as $\emptyset = \{x \mid x \neq x\}$, and set theory.

But, if this is the case, it is legitimate to ask what is the iconic mode of representation of the symbol '$\emptyset$', of the diagram representing von Neumann's Universe, i.e., the cumulative hierarchy of sets (Figure 1), and of any diagram, and well-formed formula, of the mathematics we can do in ZFC.

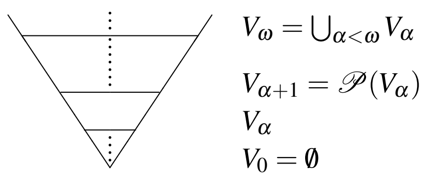

FIG. 1: *The von Neumann Universe* $V = \bigcup_{\alpha \in On} V_\alpha$

Secondly, I find puzzling Grosholz's attribution of ambiguity to the diagrams used in certain mathematical proofs. The reason for this is that, since ambiguity is a phenomenon affecting the attribution of meaning to expressions belonging to a given language, and the diagrams we use in mathematical proofs—very much like photographs and portraits—are meaningless, it follows that such diagrams cannot be ambiguous.

Indeed, as in the case of maps, their importance entirely resides on the fact that, through representing in a perspicuous manner only certain relevant features of ..., diagrams greatly increase the surveyability of such features which, for instance, happen to be conducive to the resolution of an open question we are interested in.

Thirdly, it is important to notice that no hint is given by Grosholz to explain the correlation between the purely linguistic nature of the phenomenon of ambiguity, and its mathematical productivity which manifests itself in terms of solutions found to open questions.

Trying to address the criticism above saying that linguistic and diagrammatic ambiguity make possible the interaction of different mathematical theories is unsatisfactory. For, such a reply does not explain why the interaction between different mathematical theories should be mathematically productive.

On the other hand, if we put to one side the concept of productive ambiguity, and consider mathematics to be a science of patterns, the productive interaction of two different mathematical theories T_1 and T_2 can be explained very easily.

For, if mathematical theories T_1 and T_2 generate different representations of the same object(s), they capture different aspects of the same thing(s), which, consequently, are not only compossible, but are also (probably) related to one another in the same object(s).

From this it comes as no surprise that the information about these object(s) obtained in T_2 (or T_1) might be conducive to the solution of an open question relating to them, open question formulated within T_1 (or T_2); and to the production of *Gestalt* switches (not ambiguities) between the way the object(s) is (are) represented within T_1 and the way the object(s) is (are) represented within T_2. A particularly clear, and conclusive, example of this phenomenon is the following.

For the Pythagorean mathematicians the number field consisted only of the positive integers, and of ratios of positive integers. Moreover, the positive integers were represented by them as configurations of points: they had triangular, square, etc. numbers.

When they discovered that there is no pair of positive integers m and n such that $\frac{m}{n} = \sqrt{2}$, one of the puzzles that came to occupy their minds was what to make of equations like $x = \sqrt{2}$. And since they did not have irrational numbers, the problem of solving such an equation seemed to be out of reach of their algebra (T_1).

However, with the introduction of a new system of representation—geometric algebra (T_2)—which, if $m, n \in \mathbb{Q}^+$, represents: m and n as lengths of line segments; $m+n$ as the length of the line segment obtained by concatenating along a straight line two line segments of length respectively m and n; $m \times n$ as the area of a rectangle whose sides measure respectively m and n in length; etc. it was possible to individuate—by a construction with ruler and compass, and the exploitation of some theorems of Euclidean geometry—the line segment of length x such that (α) $x^2 = m \times n$, for any two line segments of length m and n (see Figure 2).

But, now, if we put $m = 1$ and $n = 2$ in (α), it becomes apparent that, within geometric algebra, it is possible to solve the equation $x = \sqrt{2}$.

Of course, a by-product of, and not a condition for, the introduction of T_2

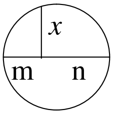

FIG. 2

alongside T_1 is that we have at least two kinds of *Gestalt* switches connected with the dawning of mathematical patterns.

The first type of *Gestalt* switch is related to the representation of $\mathbb{Z}^+$, and is expressed, for example, by the observation that the number one is represented in one system (T_1) by $\bullet$ and in the other (T_2) by a line segment of unit length —.

The second type of *Gestalt* switch consists in that, whereas for the Pythagorean mathematician, the procedure whereby we produce the diagram above is simply a way of constructing, by ruler and compass, a circle with a diameter measuring $m+n$ in length, etc.; for the geometrical algebraist, instead, the very same procedure is an algorithm which allows us to find the solution of, in particular, the equation $x = \sqrt{2}$.

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